

CENTRAL CHARGE FOR AdS_2 QUANTUM GRAVITY

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Abstract

Two-dimensional Maxwell-dilaton quantum gravity on AdS_2 with radius ℓ and a constant electric field E is studied. In conformal gauge, this is equivalent to a CFT on a strip. In order to maintain consistent boundary conditions, the usual conformal diffeomorphisms must be accompanied by a certain $U(1)$ gauge transformation. The resulting conformal transformations are generated by a twisted stress tensor, which has a central charge $c = 3kE^2\ell^4/4$ where k is the level of the $U(1)$ current. This is an AdS_2 analog of the Brown-Henneaux formula $c = 3\ell/2G$ for the central charge of quantum gravity on AdS_3 .

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1. Introduction

In a seminal 1986 paper Brown and Henneaux [1] showed that any consistent quantum theory of gravity on an asymptotically AdS_3 spacetime is a 2d CFT in the sense that the Hilbert space falls into a representation of the 2d conformal group. Expressions for the Virasoro generators were presented as integrals around the boundary at spatial infinity. They further computed the Dirac brackets of these generators and found that the central charge of the CFT is

$$c = \frac{3\ell_3}{2G_3}, \quad (1.1)$$

where ℓ_3 is the AdS_3 radius and G_3 is Newton's constant. The derivation involves only general properties of the symmetry generators and does not depend on the details of the theory. In fact an explicit unitary example of a theory to which this result applies was not found until a decade later [2].

In this paper we report a similar type of result for quantum gravity with a $U(1)$ gauge field on AdS_2 (which is conformal to a strip). Once diffeomorphisms are fixed by imposing conformal gauge for the metric, the residual symmetry group is generated by one copy of the Virasoro algebra which acts non-trivially on the boundary. We argue that the central charge is

$$c = \frac{3kE^2\ell^4}{4}, \quad (1.2)$$

where ℓ is the AdS_2 radius, E the electric field and k is the level of the current j_\pm which generates the $U(1)$. Our arguments are fairly general and relatively insensitive to the details of the theory. We do not have a clear example of a unitary theory to which this result applies, but we hope our result is a step in that direction.

The result (1.2) at first seems rather surprising as it is often said (e.g. [3,4,5]) that 2d quantum gravity, when rewritten in conformal gauge as a CFT, must have $c = 0$ because the conformal transformations are a subgroup of the diffeomorphisms which cannot consistently have a central term. What happens is this. For constant electric field, the $U(1)$ potential is singular at the boundary. This has the consequence that a conformal diffeomorphism causes the potential to violate the proper boundary condition. Hence their action on the quantum Hilbert space cannot be defined. This problem can be fixed by supplementing the conformal diffeomorphisms with a certain $U(1)$ gauge transformation generated by a current j_{\pm} . The resulting conformal transformation is generated by a twisted stress tensor

$$\tilde{T}_{\pm\pm} = T_{\pm\pm} \pm \frac{E\ell^2}{4} \partial_{\pm} j_{\pm}. \quad (1.3)$$

While the original stress tensor T of necessity has $c = 0$ the twisted stress tensor \tilde{T} has central charge given by (1.2). While a $c = 0$ CFT cannot be unitary, it is hence possible that some 2d quantum theory of gravity on AdS_2 might be recast as a unitary CFT with nonzero central charge.

This result is of potential interest for several reasons. The near-horizon geometry of every extremal black hole, including extremal Kerr, contains a universal AdS_2 factor with an electric field.¹ Currently, the black holes we understand microscopically are essentially those whose near-horizon region contains an AdS_3 factor of some kind. Hence an understanding of the near-horizon CFT with central charge (1.2) may lead to a more universal understanding of black hole entropy, perhaps in the form of entanglement entropy along the lines discussed in [6], [7].

The result is also clearly relevant to the still enigmatic AdS_2/CFT_1 correspondence. It has never been clear whether the dual “ CFT_1 ” should be thought of as conformal quantum mechanics or a chiral half of a 2d conformal theory. Both possibilities have been pursued. Although this is perhaps a matter of semantics, one way of distinguishing the two possibilities is that in the latter case one copy of the Virasoro generators acts nontrivially on the Hilbert space, while in the former only the global $SL(2, R)$ acts nontrivially. The approach of this paper is consistent with the idea that there is a nontrivial action of Virasoro. In conformal gauge, 2d quantum gravity on AdS_2 is equivalent order by order

¹ For Kerr the electric field lies in the $SU(2)$ arising from dimensional reduction of the horizon S^2 .

in perturbation theory to a 2d CFT on a strip (which admits one Virasoro action). Hence AdS_2/CFT_1 duality becomes some kind of CFT_2/CFT_2 duality on the strip.

The 2d relation (1.2) and the 3d relation (1.1) are likely directly related by dimensional reduction. Indeed, under appropriate S^1 compactification of AdS_3 to AdS_2 , the unbroken 3d conformal transformations map to twisted 2d conformal transformations precisely as indicated by (1.3)[8].² This observation motivated the present work. However we have not succeeded in making this relation precise. It would be of great interest to do so. Part of the problem is that the appropriate S^1 reduction is null at the boundary of AdS_3 , so that discrete light cone quantization must be considered.

The central charge in AdS_2 with a linear dilaton background was computed and related to 2d black holes in [12,13,14]. In that case the central charge is due to the breaking of $SL(2, R)$ by the linear dilaton. Here we consider a constant dilaton and find a different mechanism for a non-zero central charge.

This paper is organized as follows. In section 2 we describe a particular 2d gravity effective action, which will serve as our example, and its reduction to conformal gauge. It has an AdS_2 solution with an electric field. In section 3 we see that conformal diffeomorphisms do not respect the boundary conditions, and show how to fix it with a $U(1)$ gauge transformation. In section 4 we describe the resulting twisted CFT and compute its central charge.

2. Maxwell-dilaton gravity

2.1. An action

We wish to study 2D Maxwell-dilaton theories of gravity with an AdS_2 ground state supporting a constant electric field. While our considerations are quite general, it is pedagogically useful to have a specific example in mind. A simple example is described by the action

$$S = \frac{1}{2\pi} \int d^2t \sqrt{-g} \left(\eta \left(R + \frac{8}{\ell^2} \right) - \frac{\ell^2}{4} F^2 \right) + S_M, \quad (2.1)$$

² Alternately, as exploited in [9-11], the total space of the $U(1)$ bundle describing an electric field on AdS_2 is AdS_3 .

where potential boundary terms are ignored. S_M is a general matter action whose effects are assumed to vanish near boundary. It will prove convenient to introduce an auxiliary field f to eliminate the quadratic gauge field term. The action then becomes

$$S = \frac{1}{2\pi} \int d^2t \sqrt{-g} \left(\eta(R + \frac{8}{\ell^2}) - \frac{2}{\ell^2} f^2 + f \epsilon^{\mu\nu} F_{\mu\nu} \right) + S_M \quad (2.2)$$

The f equation of motion sets

$$f = \frac{\ell^2}{4} \epsilon^{\mu\nu} F_{\mu\nu}. \quad (2.3)$$

2.2. The vacuum solution

This theory has an AdS_2 vacuum solution with

$$ds^2 = -\frac{\ell^2 dt^+ dt^-}{(t^+ - t^-)^2} \quad (2.4)$$

on the Poincare wedge (where $t^\pm = t \pm \sigma$),

$$\bar{R} = -\frac{8}{\ell^2}, \quad (2.5)$$

an electric field

$$\bar{F}_{+-} = 2E\epsilon_{+-}, \quad (2.6)$$

gauge field

$$\bar{A}_\pm = \frac{E\ell^2}{4\sigma}, \quad (2.7)$$

and constant scalars

$$\begin{aligned} \bar{\eta} &= \frac{E^2 \ell^4}{4}, \\ \bar{f} &= -E\ell^2. \end{aligned} \quad (2.8)$$

2.3. CFT reformulation

We wish to study the action of the 2D conformal group on this theory by recasting it in the form of a standard 2D CFT. To this end we choose conformal gauge for the metric

$$ds^2 = -e^{2\rho} dt^+ dt^-, \quad (2.9)$$

and Lorentz gauge for the $U(1)$ potential

$$\partial_+ A_- + \partial_- A_+ = 0. \quad (2.10)$$

Locally (i.e. ignoring boundary conditions) (2.9) fixes the coordinate system up to residual conformal diffeomorphisms generated by $(\zeta^+(t^+), \zeta^-(t^-))$, while (2.10) fixes the $U(1)$ gauge up to residual transformations generated by $\theta(t^+) + \tilde{\theta}(t^-)$. In the gauge (2.10) A is determined from a scalar

$$A_{\pm} = \pm \partial_{\pm} a, \quad (2.11)$$

so that

$$F_{+-} = -2\partial_- \partial_+ a, \quad (2.12)$$

and in the background solution (2.4)-(2.8)

$$\bar{a} = \frac{E\ell^2}{2} \ln \sigma, \quad \bar{\rho} = -\ln \sigma + \ln \frac{\ell}{2}. \quad (2.13)$$

The action is then, up to total derivatives

$$S = \frac{1}{2\pi} \int d^2t \left(-4\partial_- \eta \partial_+ \rho + 4\partial_- f \partial_+ a \right. \\ \left. + \frac{4}{\ell^2} e^{2\rho} \eta - \frac{1}{\ell^2} e^{2\rho} f^2 \right) + S_M. \quad (2.14)$$

The equations of motions following from (2.14) must as usual classically be supplemented by the gauge and gravitational constraints following from the original action (2.2):

$$G_- = -2\partial_- f + j_-^M = 0, \\ G_+ = 2\partial_+ f + j_+^M = 0, \quad (2.15)$$

$$T_{--} = -2\partial_- \eta \partial_- \rho + 2\partial_- f \partial_- a - j_-^M \partial_- a + \partial_- \partial_- \eta + T_{--}^M = 0, \\ T_{++} = -2\partial_+ \eta \partial_+ \rho + 2\partial_+ f \partial_+ a + j_+^M \partial_+ a + \partial_+ \partial_+ \eta + T_{++}^M = 0. \quad (2.16)$$

Here $T_{--}^M \equiv -\frac{2\pi}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{--}}$ and $j_-^M \equiv -2\pi \frac{\delta S_M}{\delta A_+}$. These constraints, together with the residual gauge freedom, could classically be used to eliminate ρ , η , a and f as dynamical degrees of freedom, leaving only the matter fields as local degrees of freedom. The equations of motion obtained by varying the fields in (2.14), including the ρ equation $T_{+-} = 0$, imply the conservation laws³

$$\partial_+ T_{--} = 0, \quad (2.17)$$

$$\partial_- G_+ = \partial_+ G_- = 0. \quad (2.18)$$

³ In order for $T_{\pm\pm}$ to be holomorphically conserved without use of the constraints a multiple of the latter must be added to the variation of the action with respect to g^{--} as in (2.16). In deriving holomorphic G conservation we have used $2\pi \frac{\partial S_M}{\partial a} = \partial_+ j_-^M - \partial_- j_+^M$.

3. Boundary conditions and modified conformal transformations

In order to define the theory we must impose boundary conditions at $\sigma = 0$. First, we wish to restrict the diffeomorphisms so that the boundary remains at $\sigma = 0$. This requires

$$\zeta^+(t, 0) = \zeta^-(t, 0). \quad (3.1)$$

The absence of charged current flow out of the boundary requires

$$\partial_t f|_{\sigma=0} = 0. \quad (3.2)$$

A well-defined variational principle then requires

$$\partial_t a|_{\sigma=0} = A_\sigma|_{\sigma=0} = 0, \quad (3.3)$$

consistent with the background solution (2.13).

A subtlety arises because the vacuum solution (2.13) diverges on the boundary $\sigma = 0$: the conformal transformations (3.1) fail to preserve the boundary condition (3.3). At the boundary $\partial_+^n \zeta^+ = \partial_-^n \zeta^-$. It follows that near the boundary,

$$\zeta^+(t + \sigma) - \zeta^-(t - \sigma) = 2\sigma \partial_- \zeta^-(t, 0) + \mathcal{O}(\sigma^3). \quad (3.4)$$

Naively, the RHS of (3.4) can be neglected at the boundary. However since \bar{A}_\pm diverges we must be careful in taking the $\sigma \rightarrow 0$ limit. One finds the Lie derivative acts as

$$[\mathcal{L}_\zeta(\bar{A}_+ - \bar{A}_-)]_{\sigma=0} = \partial_+(\zeta^+ \bar{A}_+) + \zeta^- \partial_- \bar{A}_+ - \partial_-(\zeta^- \bar{A}_-) - \zeta^+ \partial_+ \bar{A}_- = \frac{E\ell^2}{2} \partial_+^2 \zeta^+(t, 0). \quad (3.5)$$

In order to fix this, a diffeomorphism must be accompanied by a gauge transformation $\theta(t^+) + \tilde{\theta}(t^-)$ with

$$\theta(t^+) = -\frac{E\ell^2}{4} \partial_+ \zeta^+, \quad \tilde{\theta}(t^-) = \frac{E\ell^2}{4} \partial_- \zeta^-. \quad (3.6)$$

The conformal symmetry group of the theory is hence the conformal diffeomorphisms (3.1) supplemented by the gauge transformations (3.6). For example acting on the gauge potential we have

$$\delta_\zeta A = \mathcal{L}_\zeta A - \frac{E\ell^2}{4} d(\partial_+ \zeta^+ - \partial_- \zeta^-). \quad (3.7)$$

4. The twisted CFT

4.1. Improving the stress tensor

Conformal diffeomorphisms are generated via Dirac brackets with line integrals of the current $\frac{1}{2\pi}(T_{++}\zeta^+, T_{--}\zeta^-)$. For the fields explicitly written in (2.14) these are, at fixed t^+

$$[\partial_- \rho(s^-), \partial_- \eta(t^-)] = \pi \partial_- \delta(s^- - t^-), \quad [\partial_- a(s^-), \partial_- f(t^-)] = -\pi \partial_- \delta(s^- - t^-). \quad (4.1)$$

If the equations of motion imply that the current j which generates gauge transformations is holomorphically conserved

$$\partial_- j_+ = \partial_+ j_- = 0, \quad (4.2)$$

then gauge transformations are generated via Dirac brackets with line integrals of the current $\frac{1}{2\pi}(\theta j_+, \tilde{\theta} j_-)$, and the modified conformal transformations (3.7) are generated by

$$\tilde{L}(\zeta^-) = \frac{1}{2\pi} \int dt^- \tilde{T}_{--} \zeta^-, \quad (4.3)$$

$$\tilde{L}(\zeta^+) = \frac{1}{2\pi} \int dt^+ \tilde{T}_{++} \zeta^+, \quad (4.4)$$

with $\tilde{T}_{\pm\pm}$ the improved stress tensor

$$\tilde{T}_{\pm\pm} = T_{\pm\pm} \pm \frac{E\ell^2}{4} \partial_{\pm} j_{\pm}. \quad (4.5)$$

4.2. Central charge

Let us now compute the central charge of the modified conformal transformation. The unmodified diffeomorphisms must be an anomaly-free gauge symmetry. Therefore the Dirac bracket

$$[T_{--}(t^-), T_{--}(s^-)] = -4\pi \partial_- \delta(t^- - s^-) T_{--}(s^-) + 2\pi \delta(t^- - s^-) \partial_- T_{--}(s^-) \quad (4.6)$$

has no central term. On the other hand the Dirac bracket of two holomorphically conserved currents can have one

$$[j_-(t^-), j_-(s^-)] = -2\pi k \partial_- \delta(t^- - s^-), \quad (4.7)$$

which implies that the current itself is not gauge invariant

$$\delta_\theta j_- = \frac{1}{2\pi} \left[\int \theta j, j_- \right] = k \partial_- \theta. \quad (4.8)$$

The classic example of this behavior is the Schwinger model [15,16,17]. In the fermionic formulation, the one loop diagram leads to both a central term in the commutator, as well as an anomaly in the chiral $U(1)$ current. A holomorphically conserved current of the type needed in (4.5) still exists but it is not gauge invariant. The anomaly and central term become classical and appear at the level of classical Dirac brackets in the bosonized Schwinger model.

Using (4.8) one then finds from the definition (4.5) that

$$\begin{aligned} [\tilde{T}_{--}(t^-), \tilde{T}_{--}(s^-)] &= -4\pi \partial_- \delta(t^- - s^-) \tilde{T}_{--}(s^-) + 2\pi \delta(t^- - s^-) \partial_- \tilde{T}_{--}(s^-) \\ &+ \frac{\pi k E^2 \ell^4}{8} \partial_-^3 \delta(t^- - s^-). \end{aligned} \quad (4.9)$$

Hence the modified conformal transformations have a central charge

$$c = \frac{3k E^2 \ell^4}{4}. \quad (4.10)$$

This is our main result.

4.3. Boundary energy conservation

Expanding around the background (2.4)-(2.8) with fluctuating fields $\tilde{\rho}$, \tilde{a} , \tilde{f} , $\tilde{\eta}$, the linearized equations of motion are

$$\begin{aligned} \partial_+ \partial_- \tilde{\rho} &= -\frac{1}{2\sigma^2} \tilde{\rho} \\ \partial_+ \partial_- \tilde{\eta} &= -\frac{1}{2\sigma^2} \left(\tilde{\eta} + \frac{E\ell^2}{2} \tilde{f} \right) \\ \partial_+ \partial_- \tilde{f} &= 0 \\ \partial_+ \partial_- \tilde{a} &= \frac{1}{4\sigma^2} \left(E\ell^2 \tilde{\rho} - \frac{1}{2} \tilde{f} \right). \end{aligned} \quad (4.11)$$

The combination $\tilde{\eta} + \frac{E\ell^2}{2} \tilde{f}$ behaves like a massive free field with $m^2 \ell^2 = 8$. The normalizable solution falls off near the boundary as

$$\tilde{\eta} + \frac{E\ell^2}{2} \tilde{f} \sim \sigma^2. \quad (4.12)$$

This in turn implies

$$\partial_t \partial_\sigma [\tilde{\eta} + \frac{E\ell^2}{2} \tilde{f}]_{\sigma=0} = 0. \quad (4.13)$$

Conformal invariance requires that there be no momentum flow across the boundary at $\sigma = 0$, i.e.

$$\tilde{T}_{++}(t, 0) - \tilde{T}_{--}(t, 0) = \tilde{T}_{t\sigma}(t, 0) = 0. \quad (4.14)$$

This becomes

$$\tilde{T}_{t\sigma} = -\partial_t \eta \partial_\sigma \tilde{\rho} - \partial_t \tilde{\rho} \partial_\sigma \eta + \partial_t f \partial_\sigma \tilde{a} + \partial_t \tilde{a} \partial_\sigma f + \frac{1}{\sigma} \partial_t \eta + \frac{E\ell^2}{2\sigma} \partial_t f + \partial_\sigma \partial_t \eta + \frac{E\ell^2}{2} \partial_\sigma \partial_t f. \quad (4.15)$$

Near the boundary it follows from (4.13) that

$$\tilde{T}_{t\sigma}(t, 0) = 0 \quad (4.16)$$

as required. Note that $T_{t\sigma}(t, 0) \neq 0$.

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